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METHOD OF INTEGRAL EQUATIONS IN 2D AND 3D PROBLEMS OF PLATE IMPACT ON A FLUID OF FINITE DEPTH

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This paper describes a method of integral equations for solution of the 2D and 3D problems of plate impact on an incompressible fluid of finite depth. The solutions of the equations obtained are investigated analytically and numerically. The behavior of the impact impulse is studied for various fluid depths and aspect ratios of the plate.

Introduction. 2D problems of plate impact on a fluid of finite and infinite depth have been studied in detail. A review of the results obtained is presented in [1]. The problems considered include the cases of central and noncentral impacts of a circular disk on a fluid of finite depth, vertical impact on an elliptic cylinder floating on the surface of an ideal incompressible fluid filling a confocal channel, and impacts of spheres and ellipsoids [1]. For 3D impact problems, results were obtained only for particular cases of canonical body shapes where the method of variable separation can be used. Allowance for real body shapes necessitates further development of 3D impact theory.

In the present paper, we propose a method of integral equations to solve both 2D and 3D problems of plate impact on a fluid of finite depth. The integral equations of the 2D problem are solved approximately by expanding the sought-for function and the kernel of the equation in power series in parameters related to the fluid depth. Numerical solutions of the integral equation are obtained by mechanical quadratures with evaluation of singular integrals by the Chebyshev formulas and by the collocation method, in which the sought-for potential is represented by a trigonometric series that takes into account the potential behavior at the plate edges. The results obtained are compared with the exact solution of by Keldysh [2].

The integral equation is derived for the 3D problem. It is shown that for an infinitely deep fluid, this equation coincides (with accuracy up to a factor of 1/2) with the equation used to calculate the added masses of a finite-span wing.

When the plate length is much longer than its width, the integral equation is solved exactly, which yields the distribution of the impact impulse along the plate and the total pressure impulse. These results agree with those obtained in 2D theory and in the theory of a finite-span wing.

A simplified integral equation, whose analog is available in wing theory [3], is obtained for a rectangular plate. As in the 2D case, an approximate solution of the equation is derived in the form of a power series in the reverse of the fluid depth. A numerical solution of this equation is obtained by the collocation method, in which the sought-for potential is represented by a double trigonometric series that takes into account the potential behavior at the plate edges. The coefficients of this series, the distribution of the impact impulse over the plate, and the total pressure impulse acting on a plate with various aspect ratios are calculated numerically. As the aspect ratio increases, the effect of the fluid depth becomes more pronounced.

The total impact impulse evaluated by the proposed technique is compared with the results of numerical calculations using the added masses of a rectangular wing with various aspect ratios in a infinite fluid. It is worth noting that for large aspect ratios, the calculated curve approaches the asymptotic value very slowly.

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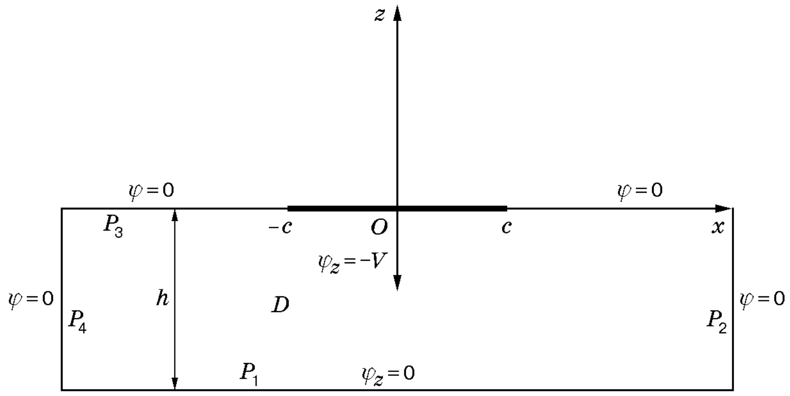

Fig. 1

1. Formulation of the Problem. We consider the 2D and 3D problems of vertical impact of a rigid flat body on the surface of an ideal incompressible fluid with density ρ. A body S of infinitely small thickness bounded by contour C floats on the surface of a fluid having depth h. The body and fluid are assumed to be at rest before the impact. At the instance of impact, the body velocity V_0 is directed normally to the free surface. We seek the fluid velocity distribution, the body velocity V after the impact, and the distribution of the pressure impulse over the body surface. The deformation of the free surface during the impact is neglected, i.e., the free surface coincides with the surface of the fluid at rest. The fluid flow near the body is assumed to be attached.

Let us introduce Cartesian coordinates $Oxyz$ with the x axis directed along the free surface and the z axis directed normally to the free surface as shown in Fig. 1. Since the fluid motion starts from the state of rest, it can be considered potential. The fluid velocity potential just after the impact is denoted by $\varphi(\boldsymbol{x})$ ($\boldsymbol{x} = (x, y, z)$). This function satisfies the Laplace equation $\Delta\varphi = 0$ in the fluid domain ($z < 0$).

The boundary conditions are written as follows:

— $\varphi_z = 0$ ($z = -h$) (nonpenetration conditions for the bottom);
— $\varphi_z = -V$ ($z = 0$) for $(x, y) \in S$ (nonpenetration conditions for the body surface);
— $\varphi = 0$ for $(x, y) \notin S$ (conditions at the free surface outside the body);
— $\varphi \to 0$ as $|\boldsymbol{x}| \to \infty$ (conditions at infinity).

Once the fluid potential is determined, we can find the pressure distribution over the body surface and the total force acting on the body. Let us denote the impulse of pressure forces over the body surface as J. Then, the Cauchy–Lagrange integral for the impact yields $J = -\rho\varphi(x, y, 0)$. The total force acting on the body is

$$P = -\rho \int\limits_S \varphi(x, y)\, dx\, dy.$$

2. 2D Problem of Vertical Impact of a Plate on a Fluid of Finite Depth. *Green's Function.* Let us introduce the Green's function $G(x, z; \xi, \zeta)$ for the domain $D = \{-\infty < x < \infty, -h < z < 0\}$ (Fig. 1). The Green's function satisfies the nonhomogeneous equation

$$\Delta G = -\delta(x - \xi)\delta(z - \zeta) \tag{1}$$

[$\delta(x)$ is the Dirac function] with the boundary conditions

$$G = 0, \quad z = \zeta = 0; \qquad G_z = 0, \quad z = -h; \qquad G_\zeta = 0, \quad \zeta = -h;$$

$$G \to 0, \qquad |x| \to \infty, \qquad |\xi| \to \infty.$$

The second Green's formula for the functions φ and G in the domain D bounded by the contour $P = \sum\limits_{i=1}^{4} P_i$ is written as

$$\varphi(x, z) = \int\limits_S \Big(G\,\frac{\partial\varphi}{\partial n} - \varphi\,\frac{\partial G}{\partial n}\Big) ds.$$

Taking into account the boundary conditions, for the points $(x,z) \in D$, we have

$$\varphi(x,z) = -\int_{-c}^{c} \varphi(\xi,0)\,\frac{\partial G}{\partial \zeta}\,(x,z;\xi,0)\,d\xi.$$

Hence, once the Green's function G is known, the potential in D is determined by its boundary values on the plate $|x| \leqslant c$.

To find the function G, we expand $\delta(z-\zeta)$ in the series [4]

$$\delta(z-\zeta) = \frac{2}{h}\sum_{m=0}^{\infty} \sin\Big(\frac{\pi}{h}\Big(m+\frac{1}{2}\Big)z\Big)\sin\Big(\frac{\pi}{h}\Big(m+\frac{1}{2}\Big)\zeta\Big).$$

The Green's function is expressed as

$$G = \frac{2}{h}\sum_{m=0}^{\infty} g_m(x,\xi)\sin(\mu_m\zeta)\sin(\mu_m z),$$

where $\mu_m = (\pi/h)(m+1/2)$. This function satisfies all boundary conditions for z and ζ. Substituting G into (1), we obtain the following equation for the functions g_m:

$$g_m'' - \mu_m^2 g = -\delta(x-\xi). \tag{2}$$

Let us substitute the integral representation of the δ-function

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ik(x-\xi))\,dk$$

into Eq. (2) and seek the functions g_m in the form

$$g_m = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ik(x-\xi))A_m(k)\,dk.$$

Substituting g_m into (2), we find that

$$A_m = \frac{1}{k^2+\mu_m^2}, \qquad g_m = \frac{1}{2\mu_m}\exp(-\mu_m|x-\xi|).$$

Here the following identity is used [5]

$$\int_0^{\infty} \frac{\cos(ax)}{x^2+b^2}\,dx = \frac{\pi}{2b}\exp(-ab) \qquad (a>0,\ \operatorname{Re} b>0).$$

Thus,

$$G(x,z;\xi,\zeta) = \frac{1}{h}\sum_{m=0}^{\infty}\frac{1}{\mu_m}\exp(-\mu_m|x-\xi|)\sin(\mu_m z)\sin(\mu_m\zeta).$$

Integral Equation. The potential on the plate is determined by solving the integral equation obtained from the condition of equal normal velocities of the plate and the fluid:

$$V = -\lim_{z\to 0}\frac{\partial\varphi}{\partial z} = \lim_{z\to 0}\int_{-c}^{c}\varphi(\xi,0)\,\frac{\partial^2 G}{\partial z\,\partial\zeta}\,(x,z;\xi,0)\,d\xi = \int_{-c}^{c}\varphi(\xi,0)\,\frac{\partial^2 G}{\partial z\,\partial\zeta}\,(x,0;\xi,0)\,d\xi, \quad |x| \leqslant c. \tag{3}$$

The kernel in (3) is defined by the series

$$\frac{\partial^2 G}{\partial z\,\partial\zeta}\,(x,0;\,\xi,0) = \frac{1}{h}\sum_{m=0}^{\infty}\mu_m\exp(-\mu_m|x-\xi|).$$

Introducing the function

$$G_1 = \sum_{m=0}^{\infty} \exp\left(-\mu_m |x-\xi|\right) = \frac{1}{2\sinh\left(\pi|x-\xi|/(2h)\right)}$$

we can write

$$\frac{\partial^2 G}{\partial z\, \partial \zeta}(x, 0;\, \xi, 0) = \frac{1}{h}\, \mathrm{sign}\,(x-\xi)\, \frac{\partial G_1}{\partial \xi}.$$

Integrating the right side of (3) by parts and taking into account that $\varphi(\pm c, 0) = 0$, we obtain

$$V = \frac{1}{2h} \int_{-c}^{c} \frac{\partial \varphi}{\partial \xi}\, \frac{1}{\sinh\left(\pi(\xi - x)/(2h)\right)}\, d\xi. \qquad (4)$$

Thus, we have a singular integral equation for the velocity on the plate, whose solution should be sought in the class of functions that are not bounded at the plate edges. It is worth noting that the integral equation derived in [6] by a different technique can be written in the same form.

Solution of the Singular Integral Equation in the Form of a Power Series. Let us seek a solution of (4) in the form of a power series in the parameter $\tau = c/h$ [7]. Introducing the new variables

$$x \to cx, \qquad \xi \to c\xi, \qquad \varphi \to cV\varphi, \qquad \frac{\partial \varphi(\xi, 0)}{\partial \xi} \to \gamma(\xi)$$

we write Eq. (4) in nondimensional form:

$$\frac{\tau}{2} \int_{-1}^{1} \gamma(\xi)\, \frac{1}{\sinh\left(\pi\tau(\xi - x)/2\right)}\, d\xi = 1. \qquad (5)$$

To solve (5), we use the expansion

$$\frac{1}{\sinh X} = \frac{1}{X} - \frac{X}{6} + \frac{7X^3}{360} + \ldots \qquad (X^2 < \pi^2). \qquad (6)$$

The solution of (5) is sought in the form

$$\gamma = \gamma_0 + \tau^2 \gamma_1 + \tau^4 \gamma_2 + \ldots. \qquad (7)$$

Substituting expansions (6) and (7) in the integral equation (5) and equating the coefficients at identical exponents of τ on both sides of the equation, we obtain the following system of equations for the functions $\gamma_0, \gamma_1, \ldots$:

$$\frac{1}{\pi} \int_{-1}^{1} \gamma_0(\xi)\, \frac{d\xi}{\xi - x} = 1, \qquad \frac{1}{\pi} \int_{-1}^{1} \gamma_1(\xi)\, \frac{d\xi}{\xi - x} = \frac{\pi}{24} \int_{-1}^{1} \gamma_0(\xi)(\xi - x)\, d\xi,$$

$$\frac{1}{\pi} \int_{-1}^{1} \gamma_2(\xi)\, \frac{d\xi}{\xi - x} = -\frac{7\pi^3}{5760} \int_{-1}^{1} \gamma_0(\xi)(x - \xi)^3\, d\xi + \frac{\pi}{24} \int_{-1}^{1} \gamma_1(\xi)(\xi - x)\, d\xi.$$

Similar expressions can be written for higher-order terms in expansion (7). The first of the above equations serves to find the function $\gamma_0(\xi)$ that corresponds to impact on the plate floating on the surface of an infinitely deep fluid. The remaining equations allow to find $\gamma_n(\xi)$ if the functions $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ are known.

Each unknown function is determined by solving a singular integral equation of the first kind in the form

$$\frac{1}{\pi} \int_{-1}^{1} \gamma(\xi)\, \frac{d\xi}{\xi - x} = f(x),$$

where $\gamma(\xi)$ is an unknown function, and the right side $f(x)$ is specified. The solution of this equation for the class of functions unbounded at both edges of the plate has the form

$$\gamma(x) = -\frac{1}{\pi\sqrt{1 - x^2}} \int_{-1}^{1} f(\xi)\, \sqrt{1 - \xi^2}\, \frac{d\xi}{\xi - x}.$$

Using this formula of inversion, we find

$$\gamma_0(x) = \frac{x}{\sqrt{1-x^2}}, \qquad \gamma_1(x) = \frac{\pi^2}{48}\,\frac{x}{\sqrt{1-x^2}}, \qquad \gamma_2(x) = -\frac{\pi^4 x}{2304}\,\frac{4.2x^2-2.05}{\sqrt{1-x^2}}.$$

For a unit length of the plate, the total impact impulse normalized by $\rho V c^2$ is equal to

$$P = -\int_{-1}^{1} \varphi(x,0)\,dx. \tag{8}$$

It can be written as a series in τ: $P = P_0 + \tau^2 P_1 + \tau^4 P_2 + \ldots$. Calculations yield

$$P_0 = \pi/2, \qquad P_1 = \pi^3/96, \qquad P_2 = -0.55\pi^5/2304,$$

and for the distribution of the impact impulse along the plate, we have

$$p_0(x) = \sqrt{1-x^2}, \quad p_1(x) = (\pi^2/48)\sqrt{1-x^2}, \quad p_2(x) = (\pi^4/2304)\sqrt{1-x^2}\,(0.75+1.4x^2).$$

With this aapproach we can derive a solution in the vicinity of $\tau = 0$. In the limiting case, $\tau \to 0$ we obtain results for an infinitely deep fluid. Passage to the limit of small depths ($\tau \to \infty$) is incorrect. Motion of a wing at small distance from a rigid boundary is thoroughly considered by Panchenkov [8], who proposed to introduce the parameter τ_1, which is always less than unity for $0 < \tau < \infty$. We introduce $\tau_1 = \sqrt{1/\tau^2+1} - 1/\tau$ such that $\tau_1 \to 1$ as $\tau \to \infty$ and $\tau_1 \to 0$ as $\tau \to 0$.

The kernel of the integral equation (5) can be represented as

$$\frac{1}{2h\sinh[\pi(\xi-x)/(2h)]} = \frac{1}{\pi(\xi-x)} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(\xi-x)^2+4n^2h^2},$$

where each term can be expanded in a series in τ_1

$$\frac{1}{(\xi-x)^2+4n^2h^2} = \frac{\tau_1^2}{n^2}\Big[1-\tau_1^2(\tau_1^2-2)-\frac{\tau_1^2}{n^2}(x-\xi)^2+\ldots\Big].$$

Substituting the expansions of the potential and the kernel into the integral equation (5) and equating terms at the same exponents of τ_1, we obtain the following system of singular integral equations:

$$\frac{1}{\pi}\int_{-1}^{1}\gamma_0(\xi)\,\frac{d\xi}{\xi-x} = 1, \qquad \frac{1}{\pi}\int_{-1}^{1}\gamma_1(\xi)\,\frac{d\xi}{\xi-x} = \frac{\pi}{6}\int_{-1}^{1}\gamma_0(\xi)(\xi-x)\,d\xi,$$

$$\frac{1}{\pi}\int_{-1}^{1}\gamma_2(\xi)\,\frac{d\xi}{\xi-x} = \frac{\pi}{6}\int_{-1}^{1}\gamma_1(\xi)(\xi-x)\,d\xi + \frac{\pi}{3}\int_{-1}^{1}\gamma_0(\xi)\Big[\xi-x-\frac{7\pi^2}{120}(\xi-x)^3\Big]d\xi.$$

Here, we take into account that [5]

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720}.$$

Hence, the solutions of these equations are

$$\gamma_0(x) = \frac{x}{\sqrt{1-x^2}}, \qquad \gamma_1(x) = \frac{\pi^2 x}{12\sqrt{1-x^2}},$$

$$\gamma_2(x) = \frac{\pi^2}{6}\Big(1+\frac{41\pi^2}{480}\Big)\frac{x}{\sqrt{1-x^2}} - \frac{7\pi^4 x^3}{240\sqrt{1-x^2}}.$$

The total impact impulse is given by

$$P_0 = \frac{\pi}{2}, \qquad P_1 = \frac{\pi^3}{24}, \qquad P_2 = \frac{\pi^3}{12}\Big(1+\frac{41\pi^2}{480}\Big) - \frac{7\pi^5}{640},$$

and the distributions of the impact impulse over the plate have the form

$$p_0 = \sqrt{1-x^2}, \quad p_1(x) = \frac{\pi^2}{12}\sqrt{1-x^2}, \quad p_2(x) = \frac{\pi^2}{6}\sqrt{1-x^2}\Big[1-\frac{\pi^2}{32}\Big(1+\frac{28}{15}x^2\Big)\Big].$$

Numerical Solution of the Singular Integral Equation. Mechanical Quadrature Method. Equation (5) can be solved using quadrature formulas for the singular integrals with subsequent solution of the final system of algebraic equations [9]. Let us calculate the integral using the Chebyshev formula

$$\int_{-1}^{1} \frac{u(\xi)\, d\xi}{\sqrt{1-\xi^2}} = \frac{\pi}{n} \sum_{k=1}^{n} u(\xi_k), \tag{9}$$

which takes into account that for $\xi = \pm 1$, the solution of Eq. (5) have singularities of the order of the square root of the distance to these points. Here the nodes ξ_k, which are roots of the Chebyshev polynomials of the first kind $T_n(\xi)$, are calculated from the formula

$$\xi_k = \cos\Big(\frac{2k-1}{2n}\,\pi\Big) \qquad (k = 1, 2, \ldots, n).$$

Equation (5) is to be satisfied at the points $x_i = \cos(\pi i/n)$ $(i = 1, 2, \ldots, n-1)$, which correspond to the roots of the Chebyshev polynomials of the second kind. Solution of Eq. (5) with the above-mentioned singularity should be determined with the additional condition

$$\int_{-1}^{1} \gamma(\xi)\, d\xi = 0. \tag{10}$$

The integral in Eq. (10) is calculated from the quadrature formula (9). As a result, we obtain the following system of n linear algebraic equations for n unknown quantities $\gamma(\xi_k)$ $(k = 1, 2, \ldots, n)$:

$$\sum_{k=1}^{n} \gamma(\xi_k) K(\xi_k, x_i) = 1, \quad \sum_{k=1}^{n} \gamma(\xi_k) = 0, \quad K(\xi_k, x_i) = \frac{\tau}{2\sinh\,[\pi\tau(\xi_k - x_i)/2]}. \tag{11}$$

System (11) is a discrete analog of the integral equation (5) and condition (10) and can be solved numerically using standard mathematical software. The value of the function γ at an arbitrary point ξ are determined from the Lagrange interpolation polynomial with nodes ξ_k. The total impact impulse determined from (8) after integration by parts is also calculated using (9). As is noted in [9], the convergence of the process follows from convergence of the quadrature formula (9) and uniqueness of the solution of the integral equation (5) subject to condition (10).

Collocation Technique. Let us introduce the new variables $\xi = \cos\theta$ and $x = \cos\psi$ in (5) and seek the potential on the plate in the form of the series

$$\varphi(\theta, 0) = \sum_{m=1}^{N} \frac{a_m}{m} \sin(m\theta). \tag{12}$$

The integrals obtained by substituting (12) into (5)

$$I_{im} = \int_{0}^{\pi} \frac{\cos(m\theta)}{\sinh\,[\pi\tau(\cos\theta - \cos\psi_i)/2]}\, d\theta$$

are calculated from the Chebyshev quadrature formula (9):

$$I_{im} = \frac{\pi}{n} \sum_{k=1}^{n} \frac{\cos(m\theta_k)}{\sinh\,[\pi\tau(\cos\theta_k - \cos\psi_i)/2]},$$

where $\theta_k = (2k-1)\pi/(2n)$ and $\psi_i = \pi i/n$ $(i = 1, 2, \ldots, n-1)$. The number of terms in the series N should be equal $n-1$ to ensure that the number of equations in the algebraic system coincides with the number of unknown coefficients in (12).

The total impact impulse $P = (\pi/2)a_1$ is defined by the coefficient for the singularity at the plate edge.

Numerical Results. Results of calculations of the total impact impulses P_h and P_τ obtained using the series in τ and τ_1, respectively, results of numerical solution of the integral equation by the mechanical quadrature method P_r and by the collocation method P_c, and the exact solution P_k [2] are given in Table 1. For $\bar{h} = h/c < 1$, the impulse P_k was evaluated by the approximate formula for shallow fluid depth derived in [10] from the exact solution obtained by Keldysh [2]. However, as follows from Table 1, even for $\bar{h} = 0.8$, this formula gives a large error. In the series in τ and τ_1, we took into account terms up to the sixth power. For $\bar{h} < 1$, the series in τ becomes invalid. For

TABLE 1

$\bar{h}$	P_h	P_τ	P_r	P_c	P_k
0.2	—	3.0782	4.289 853	4.289 853	4.2156
0.4	—	2.4604	2.689 338	2.689 339	2.5492
0.6	—	2.1304	2.192 123	2.192 123	1.9936
0.8	—	1.9417	1.965 002	1.965 002	1.7158
1.0	1.8207	1.8307	1.841 780	1.841 780	1.8276
3.0	1.6058	1.6058	1.605 823	1.605 822	1.6155
5.0	1.5837	1.5836	1.591 014	1.591 013	1.6154
∞	1.5708	1.5708	—	—	1.5708

TABLE 2

x	$\bar{P}_h$			$\bar{P}_\tau$			$\bar{P}_k$		
	$\bar{h}=1$	$\bar{h}=5$	$\bar{h}=\infty$	$\bar{h}=1$	$\bar{h}=5$	$\bar{h}=\infty$	$\bar{h}=1$	$\bar{h}=5$	$\bar{h}=\infty$
0	1.174	1.008	1.000	1.174	1.008	1.000	1.172	1.008	1.000
±0.2	1.148	0.988	0.980	1.144	0.988	0.980	1.146	0.993	0.980
±0.4	1.067	0.924	0.917	1.072	0.924	0.917	1.076	0.920	0.917
±0.6	0.922	0.807	0.800	0.931	0.807	0.800	0.926	0.806	0.800
±0.8	0.682	0.605	0.600	0.709	0.605	0.600	0.688	0.604	0.600
±1.0	0	0	0	0	0	0	0	0	0

the parameter τ_1, the impact impulse P_τ was calculated for rather small values of $\bar{h}$ as well. However, for $\bar{h} = 0.2$, its value differs appreciably from the exact value P_k and from the P_r obtained by solution of the integral equation. In general, the results show satisfactory agreement.

The distribution of the impact impulse normalized by $\rho V c^2$ over the plate length $\bar{P}(x)$ is presented in Table 2 ($\bar{P}_h$ and $\bar{P}_\tau$ are obtained by series expansions in τ and τ_1, respectively, and $\bar{P}_k$ are data taken from [2]). Distributions calculated by approximate methods agree well with exact ones.

3. Vertical Impact of a Finite-Span Plate on a Fluid of Finite Depth. *Green's Function of an Infinite Strip.* Let us introduce the Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ [$\boldsymbol{x} = (x, y, z)$ and $\boldsymbol{y} = (\xi, \eta, \zeta)$] for the strip $D = \{-\infty < x < \infty,\ -\infty < y < \infty,\ -h < z < 0\}$ (Fig. 2). The Green's function satisfies the nonhomogeneous equation

$$\Delta G = -\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{13}$$

with the boundary conditions

$$G = 0 \quad \text{for} \quad z = \zeta = 0, \quad G_z = 0 \quad \text{for} \quad z = -h;$$

$$G_\zeta = 0 \quad \text{for} \quad \zeta = -h, \quad G \to 0 \quad \text{for} \quad |\boldsymbol{x}| \to \infty; \quad G \to 0 \quad \text{for} \quad |\boldsymbol{y}| \to \infty.$$

As in the 2D case, the second Green's formula for the functions φ and G in the domain D is written as

$$\varphi(\boldsymbol{x}) = -\int_S \varphi(\xi, \eta, 0)\, \frac{\partial G}{\partial \zeta}\, (x; \xi, \eta, 0)\, d\xi\, d\eta.$$

Once the Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ is known, the function φ can be determined from its values on the boundary of a flat body S. Let us seek Green's function in the form

$$G = \frac{2}{h} \sum_{m=0}^{\infty} \sin(\mu_m z) \sin(\mu_m \zeta) g_m(x, y; \xi, \eta).$$

Substituting G into (13), we obtain the following equation for the functions g_m:

$$\frac{\partial^2 g_m}{\partial x^2} + \frac{\partial^2 g_m}{\partial y^2} - \mu_m^2 g_m = -\delta(x - \xi)\delta(y - \eta). \tag{14}$$

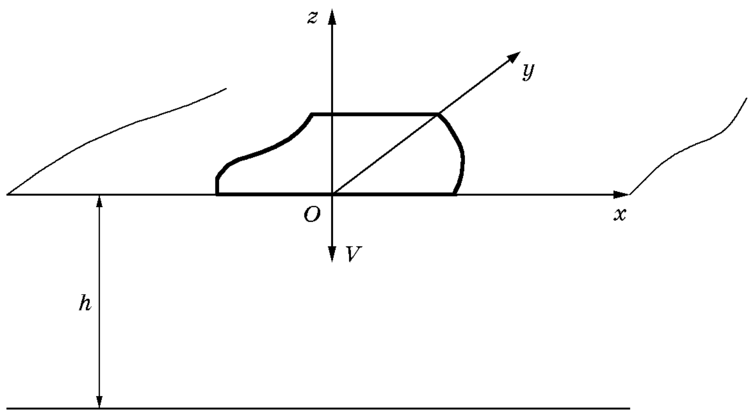

Fig. 2

Writing the δ-function and the fundamental solution of (14) in the form

$$\delta(x-\xi)\delta(y-\eta) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(ik_1(x-\xi)+ik_2(y-\eta)\right) dk_1\, dk_2,$$

$$g_m(x,y;\xi,\eta) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_m(k_1,k_2)\exp\left(ik_1(x-\xi)+ik_2(y-\eta)\right) dk_1\, dk_2$$

and substituting these expressions into (14), we obtain

$$A_m = \frac{1}{k_1^2+k_2^2+\mu_m^2}, \quad g_m = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(ik_1(x-\xi)+ik_2(y-\eta))}{k_1^2+k_2^2+\mu_m^2}\, dk_1\, dk_2. \tag{15}$$

In (15) we convert to polar coordinates (k,θ): $k_1 = k\cos\theta$ and $k_2 = k\sin\theta$. Using the identity

$$\int_0^{2\pi} \exp\left(ik\rho\cos\theta\right) d\theta = 2\pi J_0(k\rho),$$

where $J_0(k\rho)$ is a zeroth-order Bessel's function, we find that

$$g_m = \frac{1}{2\pi}\int_0^{\infty} \frac{kJ_0(k\rho)}{k^2+\mu_m^2}\, dk.$$

The expression for this integral is given in [4]. Thus,

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{\pi h}\sum_{m=0}^{\infty} K_0(\mu_m\rho)\sin\left(\mu_m z\right)\sin\left(\mu_m\zeta\right).$$

Here K_0 is a modified Bessel function and $\rho^2 = (x-\xi)^2+(y-\eta)^2$. The Green's function can be written as $G(\boldsymbol{x},\boldsymbol{y}) = G_0 - G_2$, where

$$G_0 = \frac{1}{2\pi h}\sum_{m=1}^{\infty} K_0(\nu_m\rho)[\cos\left(\nu_m(z-\zeta)\right) - \cos\left(\nu_m(z+\zeta)\right)],$$

$$G_2 = \frac{1}{2\pi h}\sum_{m=1}^{\infty} K_0(2\nu_m\rho)[\cos\left(2\nu_m(z-\zeta)\right) - \cos\left(2\nu_m(z+\zeta)\right)], \qquad \nu_m = \frac{\pi m}{2h}.$$

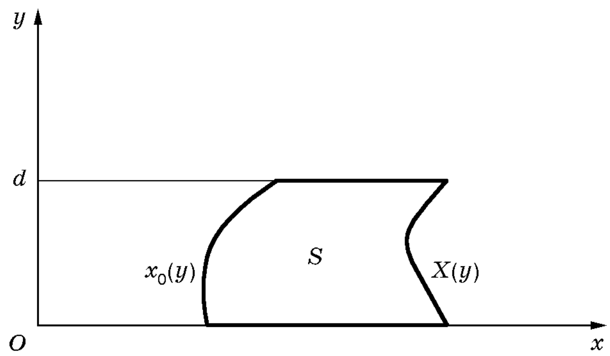

Fig. 3

The series in the expressions for G_0 and G_2 can be summed up [5]. Finally, we obtain

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\Big(\frac{1}{r_n} - \frac{1}{r'_n}\Big),$$

where $r_n^2 = (x-\xi)^2 + (y-\eta)^2 + [z-(2nh+(-1)^n\zeta)]^2$ and $r_n'^2 = (x-\xi)^2 + (y-\eta)^2 + [z-(2nh-(-1)^n\zeta)]^2$.

Integral Equation. The nonpenetration condition for the surface S allows us to formulate an integral equation for the potential at points on this surface:

$$V = -\lim_{z\to 0}\frac{\partial\varphi}{\partial z} = \lim_{z\to 0}\int\limits_S \varphi(\xi,\eta,0)\,\frac{\partial^2 G}{\partial z\,\partial\zeta}\,(\boldsymbol{x},\boldsymbol{y})\,d\xi\,d\eta = \int\limits_S \varphi(\xi,\eta,0)\,\frac{\partial^2 G}{\partial z\,\partial\zeta}\,(x,y,0;\xi,\eta,0)\,d\xi\,d\eta. \tag{16}$$

The kernel of the integral equation (16) is given by the series

$$\frac{\partial^2 G}{\partial z\,\partial\zeta}(x,y,0;\xi,\eta,0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}(-1)^n\Big(\frac{1}{R_n^3} - \frac{12n^2h^2}{R_n^5}\Big),$$

where $R_n^2 = (x-\xi)^2 + (y-\eta)^2 + 4n^2h^2$.

Let us assume that the surface has the shape shown in Fig. 3. The upper and the lower boundaries of the surface are the straight lines $y = 0$ and $y = d$. The equations of the left $[x = x_0(y)]$ and right $[x = X(y)]$ boundaries of the domain S are arbitrary continuous functions.

Integrating by parts over ξ the integral in the right side of Eq. (16) and taking into account that $\varphi(x_0(\eta),\eta,0) = \varphi(X(\eta),\eta,0) = 0$, we obtain

$$V = \frac{1}{2\pi}\int\limits_0^d\int\limits_{x_0(\eta)}^{X(\eta)}\gamma(\xi,\eta)\Big\{\frac{x-\xi}{(y-\eta)^2[(x-\xi)^2+(y-\eta)^2]^{1/2}}$$

$$+\,2\sum_{n=1}^{\infty}(-1)^n\Big[\frac{x-\xi}{Y_nR_n} - \frac{12n^2h^2}{Y_n^2}\Big(\frac{x-\xi}{R_n} - \frac{(x-\xi)^3}{3R_n^3}\Big)\Big]\Big\}d\xi\,d\eta, \tag{17}$$

where $Y_n = (y-\eta)^2 + 4n^2h^2$.

The first term in the kernel of Eq. (17) can be written as

$$\frac{x-\xi}{(y-\eta)^2[(x-\xi)^2+(y-\eta)^2]^{1/2}} = -\frac{\partial}{\partial y}\,\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(y-\eta)(x-\xi)}.$$

Then,

$$V = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int\limits_0^d\int\limits_{x_0(\eta)}^{X(\eta)}\gamma(\xi,\eta)\,\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(x-\xi)(y-\eta)}\,d\xi\,d\eta$$

$$+\,\frac{1}{\pi}\int\limits_0^d\int\limits_{x_0(\eta)}^{X(\eta)}\gamma(\xi,\eta)\sum_{n=1}^{\infty}(-1)^n\Big[\frac{x-\xi}{R_nY_n} - \frac{12n^2h^2}{Y_n^2}\Big(\frac{x-\xi}{R_n} - \frac{(x-\xi)^3}{3R_n^3}\Big)\Big]d\xi\,d\eta. \tag{18}$$

Dropping the series in n on the right sides of (17) and (18), we obtain, with accuracy up to a factor of 1/2, an integral equation for zero-circulation flow about a thin finite-span wing with surface S placed in the uniform flow with velocity V.

Thus, we have a 2D singular integral equation for the x velocity component along the plate surface. A solution of this equation should be sought in the class of functions that are not bounded at the edges $x = x_0(y)$ and $x = X(y)$ and vanish in the segments $[\,y = 0,\ x_0(0) < x < X(0)]$ and $[y = d,\ x_0(d) < x < X(d)]$.

Fluid of Infinite Depth. For a fluid of infinite depth, the integral equation (18) reduces to

$$V = -\frac{1}{2\pi}\frac{\partial}{\partial y}\int\limits_0^d\int\limits_{x_0(\eta)}^{X(\eta)}\gamma(\xi,\eta)\,\frac{\sqrt{(x-\xi)^2+(y-\eta)^2}}{(x-\xi)(y-\eta)}\,d\xi\,d\eta. \tag{19}$$

We assume that one dimension of the wing, for example, the dimension in the x direction, is large compared to the other. Using the classical approximation of finite-span wing theory, we write $\sqrt{(x-\xi)^2+(y-\eta)^2} = |x-\xi|$. Then, Eq. (19) becomes

$$V = \frac{1}{2\pi}\frac{\partial}{\partial y}\int\limits_0^d\int\limits_{x_0(\eta)}^{X(\eta)}\gamma(\xi,\eta)\,\frac{\operatorname{sign}(x-\xi)}{\eta-y}\,d\xi\,d\eta.$$

Taking into account that the potential φ vanish at the edges of the plate S, we have

$$V = -\frac{1}{\pi}\frac{\partial}{\partial y}\int\limits_0^d\frac{\varphi(x,\eta,0)}{y-\eta}\,d\eta. \tag{20}$$

Integrating (20) by parts, introducing the derivative with respect to y under the integration sign, and using the nondimensional variables $\eta \to (2\eta - d)/d$ and $y \to (2y - d)/d$, we finally obtain

$$\frac{1}{\pi}\int\limits_{-1}^{1}\frac{\partial\varphi}{\partial\eta}\,(x,\eta,0)\,\frac{1}{\eta-y}\,d\eta = V. \tag{21}$$

The solution of the integral equation (21) unbounded at the both edges of the plate has the form

$$\frac{\partial\varphi}{\partial y} = \frac{V}{\pi\sqrt{1-y^2}}\int\limits_{-1}^{1}\frac{\sqrt{1-\eta^2}}{y-\eta}\,d\eta.$$

Integration yields $\varphi(x,y) = -V\sqrt{1-y^2}$.

According to (8), the total impulse of the pressure forces acting on the plate cross section $x = \text{const}$ (per unit length in the x direction) is equal to $P = \pi/2$ and coincides with the value predicted by 2D theory, which is applicable to the case considered.

Rectangular Plate. Let the surface S be a rectangle $[-c \leqslant x \leqslant c,\ -l \leqslant y \leqslant l]$. We note that

$$\frac{\partial}{\partial y} = -\frac{\partial}{\partial\eta}, \qquad \frac{x-\xi}{R_nY_n} = -\frac{1}{2nh}\frac{\partial}{\partial\eta}\arctan\frac{(x-\xi)(y-\eta)}{2nhR_n},$$

$$\frac{x-\xi}{R_nY_n^2} = -\frac{1}{8n^2h^2(x-\xi)}\frac{\partial}{\partial\eta}\frac{R_n(y-\eta)}{Y_n} - \frac{(x-\xi)^2-4n^2h^2}{16n^3h^3(x-\xi)^2}\frac{\partial}{\partial\eta}\arctan\frac{(x-\xi)(y-\eta)}{2nhR_n},$$

$$\frac{4n^2h^2(x-\xi)^3}{R_n^3Y_n^2} = -\frac{\partial}{\partial\eta}\Big\{\frac{(x-\xi)(y-\eta)}{2R_nY_n} - \frac{[(x-\xi)^2-12n^2h^2](y-\eta)}{2(x-\xi)R_n[(x-\xi)^2+4n^2h^2]}$$
$$+\frac{(x-\xi)(y-\eta)}{R_n[(x-\xi)^2+4n^2h^2]} + \frac{(x-\xi)^2-12n^2h^2}{4nh(x-\xi)^2}\arctan\frac{(x-\xi)(y-\eta)}{2nhR_n}\Big\}.$$

Substituting these expressions into (18), integrating by parts over η, and taking into account that $\gamma(\xi,\pm l) = 0$, we obtain the integral equation

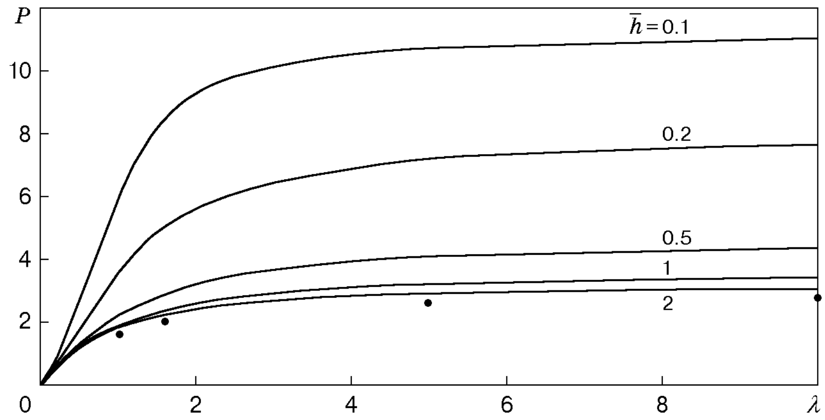

Fig. 4

$$V = -\frac{1}{2\pi} \int\limits_{-c}^{c} \int\limits_{-l}^{l} \frac{\partial \gamma}{\partial \eta}(\xi, \eta) \Bigg\{ \frac{\sqrt{(x-\xi)^2 + (y-\eta)^2}}{(x-\xi)(y-\eta)}$$

$$+ 2 \sum_{n=1}^{\infty} (-1)^n \frac{(x-\xi)(y-\eta)(R_n^2 + 4n^2h^2)}{[(x-\xi)^2 + 4n^2h^2][(y-\eta)^2 + 4n^2h^2]R_n} \Bigg\} d\xi\, d\eta. \qquad (22)$$

The solution of Eq. (19) in the form of a series in τ is written as $\gamma = \gamma_0 + \tau^3\gamma_1 + \tau^6\gamma_2 + \ldots$ because in the series in τ for the part of the kernel that takes into account the effect of the bottom, the first term is equal to $0.45077(x-\xi)(y-\eta)\tau^3$. Here allowance is made for the fact that [5]

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} = -0.90154.$$

This result differs from that obtained in the 2D case where the solution was expanded in a series in τ^2.

Numerical Solution of Eq. (22) *by the Collocation Method.* Let us introduce the new variables $x = c\cos\theta_1$, $y = l\cos\psi_1$, $\xi = c\cos\theta$, $\eta = l\cos\psi$. The potential on the plate surface is assumed to have the form

$$\varphi(\xi, \eta) = \sum_{m=1}^{M} \sum_{k=1}^{N} \frac{a_{km}}{km} \sin(k\theta) \sin(m\psi), \qquad (\xi, \eta) \in S. \qquad (23)$$

In the nondimensional variables $\varphi \to cV\varphi$, $x \to cx$, $\xi \to c\xi$, $y \to ly$, and $\eta \to l\eta$, the integral equation (22) reduces to

$$2\pi\lambda = -\int\limits_{-1}^{1} \int\limits_{-1}^{1} \frac{\partial \gamma}{\partial \eta} \Bigg\{ \frac{\sqrt{(x-\xi)^2 + \lambda^2(y-\eta)^2}}{(x-\xi)(y-\eta)}$$

$$+ 2\lambda^2 \sum_{n=1}^{\infty} (-1)^n \frac{(x-\xi)(y-\eta)(R_n^2 + 4n^2\tau^{-2})}{[(x-\xi)^2 + 4n^2\tau^{-2}][\lambda^2(y-\eta)^2 + 4n^2\tau^{-2}]R_n} \Bigg\} d\xi\, d\eta, \qquad (24)$$

where $\lambda = l/c$ is the aspect ratio of the plate.

Substitution of series (23) into Eq. (24) and numerical integration by the Chebyshev formula give the following system of linear algebraic equations:

$$\sum_{k=1}^{N} \sum_{m=1}^{M} a_{km} I_{ij}^{km} = 2\pi\lambda.$$

Here

$$I_{ij}^{km} = -\frac{\pi^2}{NM} \sum_{p=1}^{N+1} \sum_{q=1}^{M+1} \cos(k\theta_p) \cos(m\psi_q) K(\omega_i, \chi_j; \theta_p, \psi_q),$$

$K(x, y; \xi, \eta)$ is the kernel of Eq. (24), $\omega_i = \pi i/(N+1)$, $\chi_j = \pi j/(M+1)$ ($i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$) are the collocation points, $\theta_p = (2p-1)\pi/(2(N+1))$, and $\psi_q = (2q-1)\pi/(2(M+1))$ are the nodes of the Cnebyshev formula applied to the repeated integral that replaces the double integral in Eq. (24).

The total impact impulse normalized by $\rho V c^2 l$ is

$$P = -\int_{-1}^{1}\int_{-1}^{1} \xi\eta \,\frac{\partial^2\varphi}{\partial\xi\partial\eta}\, d\xi\, d\eta = -\frac{\pi^2}{4}\, a_{11}.$$

Numerical Results. Dependences of the total impact impulse P on the aspect ratio of the plate λ are shown in Fig. 4 for various values of the non-dimensional fluid depth $\bar{h}$. Numerical solution of the integral equation (24) was performed by the collocation method. The points Fig. 4 show values of the impulse P obtained using the added mass coefficients for rectangular wings in an infinite fluid [11]. One can see that these values are close to the curve for the impact impulse at $\bar{h} = 2$. As the aspect ratio increases, the effect of the fluid depth becomes more pronounced. For fixed $\bar{h}$, a decrease in aspect ratio (due to a decrease in plate width) attenuates the influence of the boundary (bottom) on the impact impulse acting on the plate. In addition, it is worth noting that for large aspect ratios, the curves obtained for various $\bar{h}$ approach the asymptotic values rather slowly.

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